Short Papers

Eigenmodes in a Toroidal Cavity of Elliptic Cross Section

M. S. Janaki and B. Dasgupta

Abstract—**The axisymmetric electromagnetic eigenmodes of a toroidal resonator with elliptic cross section are analyzed by solving the basic equations up to the lowest order in inverse aspect ratio. The effects of toroidicity and ellipticity of the cross section are quite significant for both TE and TM modes.**

I. Introduction

The eigenmodes of an electromagnetic wave propagating in a toroidal cavity are of interest in radio frequency (RF) breakdown, initiation of plasma discharge, and RF heating of tokamak plasmas. The determination of such eigenmodes is difficult owing to the non-separability of the boundary value problem in toroidal coordinates. The modes of toroidal resonators with circular cross section have been investigated in several papers by approximate analytical [1]–[4] or purely numerical methods [5].

Toroidal devices with noncircular cross sections have been assumed to be advantageous for high beta fusion reactors. Eigenmode calculations for empty toroidal resonators with arbitrary cross section have been carried out through semianalytical methods [6]. Collocation methods have been used by Cap [7] to satisfy the boundary conditions on toroidal surfaces of arbitrary cross section by solving the Helmholtz equation in circular cylindrical coordinates. Variational techniques have been employed by Wu *et al.* [8] to study resonant frequencies in a torus with elliptical cross section. In this paper we have obtained analytical solutions for resonant cavity modes in toroidal configurations with elliptic cross sections by making a perturbation expansion in inverse aspect ratio.

II. Toroidal Elliptic Geometry

Two different configurations of a torus with an elliptic cross section are possible, depending on whether $a/b > 1$ (oblate elliptic torus) or $b/a > 1$ (prolate elliptic torus), where a and b are the semimajor and minor axes, respectively, of the ellipse. Fig. 1 shows a three-dimensional toroidal cavity and a typical poloidal cut of a torus with a prolate elliptic cross section $(b/a > 1)$.

CC' is a straight line of length $2R_0$ lying in the $x-y$ plane making an angle ϕ with the x-axis, and R_0 is the major radius of the torus. The coordinates of the point P are defined by η, $\psi = \angle PCA$ and ϕ, the toroidal angle. The surface of the torus is defined by $\eta = \eta_0$. For fixed η and ϕ, the angle ψ varies from 0 to 2π in passing once around the ellipse, and η varies from 0 along the line joining the foci of the ellipse to η_0 on the surface of the torus. The circle generated by CC' rotating about the z-axis is the circle to which the torus collapses when both the major and minor axes of the ellipse are reduced to zero.

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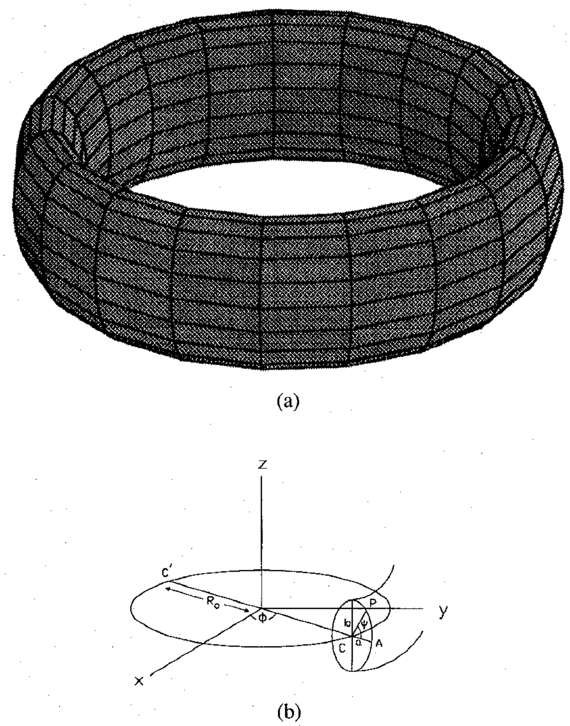


Fig. 1. (a) Toroidal cavity with prolate elliptic cross section. (b) Poloidal cut of the torus.

A. Prolate Elliptic Cross Section

The transformation relations [9] between the Cartesian coordinates and the prolate elliptical coordinates (η, ψ, ϕ) are defined by

$$x = (R_0 + \alpha \sinh\eta \cos\psi)\cos\phi,$$
$$y = (R_0 + \alpha \sinh\eta \cos\psi)\sin\phi,$$
$$z = \alpha \cosh\eta \sin\psi \tag{1}$$

where α is a constant defined by

$$\alpha \sinh\eta_0 = a, \qquad \alpha \cosh\eta_0 = b, \quad \text{and} \quad \alpha = \sqrt{b^2 - a^2}.$$

The inverse aspect ratio of the torus ϵ and the elongation of the elliptic cross section e are defined by

$$\epsilon = \frac{a}{R_0} = \frac{\alpha \sinh\eta_0}{R_0}, \qquad e = \frac{b}{a} = \cosh\eta_0.$$

B. Oblate Elliptic Cross Section

For this case we introduce the following transformations:

$$x = (R_0 + \alpha \cosh\eta \sin\psi)\cos\phi,$$
$$y = (R_0 + \alpha \cosh\eta \sin\psi)\sin\phi,$$
$$z = \alpha \sinh\eta \cos\psi. \tag{2}$$

Here

$$\alpha \cosh\eta_0 = a, \qquad \alpha \sinh\eta_0 = b,$$

$$\alpha = \sqrt{a^2 - b^2} \quad \text{and} \quad \epsilon = \alpha \cosh\eta_0 / R_0.$$

III. TE AND TM MODES

In a toroidal geometry there are no strictly E-transverse (TE) or H-transverse (TM) fields. An important exception is the toroidally uniform case where the solutions are independent of toroidal angle ϕ. In this axisymmetric case, electromagnetic modes are transverse either in electric or in magnetic fields and can be derived from an electric or a magnetic Hertz vector having only a toroidal or ϕ-component. Fields of these two types can be found by inspection of Maxwell's equations written in the coordinate system defined by (1) and (2) and putting $\partial/\partial\phi = 0$.

TM modes:

$$\begin{aligned} E_\phi &= U/\sqrt{h_3}, \\ -i\frac{\omega}{c}B_\eta &= \frac{1}{h_2 h_3}\frac{\partial}{\partial\psi}(\sqrt{h_3}U), \\ i\frac{\omega}{c}B_\psi &= \frac{1}{h_3 h_1}\frac{\partial}{\partial\eta}(\sqrt{h_3}U). \end{aligned} \tag{3}$$

TE modes:

$$\begin{aligned} B_\phi &= U/\sqrt{h_3}, \\ i\frac{\omega}{c}E_\eta &= \frac{1}{h_2 h_3}\frac{\partial}{\partial\psi}(\sqrt{h_3}U), \\ -i\frac{\omega}{c}E_\psi &= \frac{1}{h_3 h_1}\frac{\partial}{\partial\eta}(\sqrt{h_3}U) \end{aligned} \tag{4}$$

where

$$h_1 = h_2 = \alpha\sqrt{\cosh^2\eta - \sin^2\psi}$$

while $h_3 = R_0 + \alpha \sinh\eta\cos\psi$ for prolate elliptical coordinates and $h_3 = R_0 + \alpha\cosh\eta\sin\psi$ for oblate elliptical coordinates.

The above equations have been divided into two groups, (3) containing only (E_ϕ, B_η and B_ψ) components and (4) containing only (B_ϕ, E_η and E_ψ) components. Therefore, there exist two types of solutions, one with $B_\phi = 0$, which in analogy to the corresponding cylindrical case is called the TM mode, and one with $E_\phi = 0$, which is called the TE mode. The potential U occuring in (3) and (4) satisfies the following differential equation in prolate elliptical coordinates:

$$\frac{\partial^2 U}{\partial\eta^2} + \frac{\partial^2 U}{\partial\psi^2} + \alpha^2\left[k^2 - \frac{3/4R_0^2}{\left(1+\epsilon\frac{\sinh\eta}{\sinh\eta_0}\cos\psi\right)^2}\right] \cdot (\cosh^2\eta - \sin^2\psi)U = 0 \tag{5a}$$

and for the oblate elliptical coordinates defined by (2)

$$\frac{\partial^2 U}{\partial\eta^2} + \frac{\partial^2 U}{\partial\psi^2} + \alpha^2\left[k^2 - \frac{3/4R_0^2}{\left(1+\epsilon\frac{\cosh\eta}{\cosh\eta_0}\sin\psi\right)^2}\right] \cdot (\cosh^2\eta - \sin^2\psi)U = 0 \tag{5b}$$

with $k = \omega/c$. We solve (5) by introducing an expansion in the inverse aspect ratio, and for both types of cross section we write

$$U = \sum_n \epsilon^n U_n.$$

The zero-order equation is obtained as

$$\frac{\partial^2 U_0}{\partial\eta^2} + \frac{\partial^2 U_0}{\partial\psi^2} + 4q(\cosh^2\eta - \sin^2\psi)U_0 = 0 \tag{6}$$

where $4q = \alpha^2[k^2 - 3/4R_0^2]$. Here, α takes on the appropriate expression for the two different types of cross section. The appropriate formal solution of (6) in terms of Mathieu functions [10] is

$$U_0 = \sum_{m=0}^{\infty} C_m Ce_m(\eta, q)ce_m(\psi, -q) + \sum_{m=1}^{\infty} S_m Se_m(\eta, q)se_m(\psi, -q). \tag{7}$$

For any given value of m, there are two kinds of solutions, namely, the even and odd modes

$$U_0(\eta, \psi) = C_m Ce_m(\eta, q)ce_m(\psi, -q), \qquad m = 0, 1, 2 \tag{8}$$

$$U_0(\eta, \psi) = S_m Se_m(\eta, q)se_m(\psi, -q), \qquad m = 1, 2 \tag{9}$$

where C_m and S_m are arbitrary constants. The notation $ce_m(\psi, q)$ signifies a cosine type of Mathieu function of order m which reduces to a multiple of $\cos m\psi$ when $q = 0$. Since m is any positive integer, there are an infinite number of such solutions. Similarly, $se_m(\psi, q)$ signifies a sine type of Mathieu function. $Ce_m(\eta, q)$ and $Se_m(\eta, q)$ define modified Mathieu functions of integer order which reduce, respectively, to $\cosh m\eta$ or $\sinh m\eta$ when $q = 0$.

In the limit $\eta \to \infty$ and $\alpha \to 0$, such that $\alpha\cosh\eta \to \alpha\sinh\eta \to a$, the ellipse of semimajor axis a tends to a circle of radius a.

In this limit

$$ce_m(\psi, q) \to \cos m\psi \tag{10}$$

and

$$Ce_m(\eta, q) \to J_m\left(\sqrt{(k^2R_0^2 - 3/4)}a/R_0\right) \tag{11}$$

so that the solution of (6) for a torus of circular cross section obtained through the limiting procedure is

$$U_0 = J_m\left(\sqrt{(k^2R_0^2 - 3/4)}a/R_0\right)\cos m\psi. \tag{12}$$

The solution given in (12) corresponds exactly to that obtained in [11] as the zero-order solutions of Helmholtz equation in quasitoroidal coordinates.

IV. DETERMINATION OF EIGENFREQUENCIES

In order to obtain the eigenfrequencies, we have to impose the boundary conditions that the tangential component of the electric field $E_{\tan}$ vanishes on the surface $\eta = \eta_0$ of the perfectly conducting elliptic torus. This requirement is equivalent to

$$U(\eta_0) = 0 \text{ for the TM modes} \tag{13a}$$

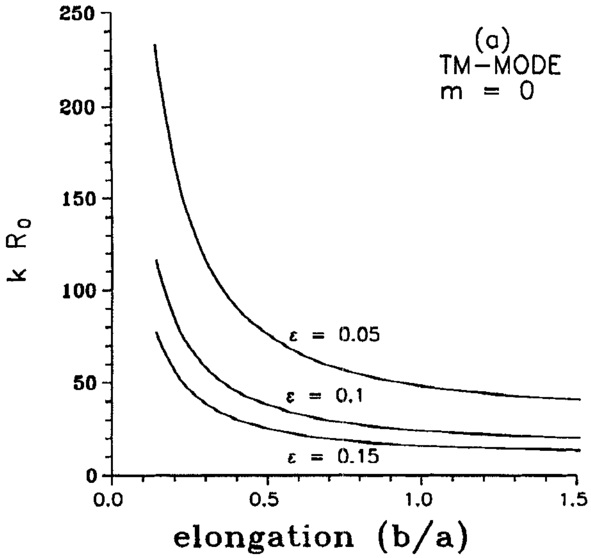
$$\left.\frac{\partial}{\partial\eta}\sqrt{h_3}U(\eta)\right|_{\eta=\eta_0} = 0 \text{ for the TE modes.} \tag{13b}$$

The boundary conditions satisfied up to the lowest order in inverse aspect ratio require

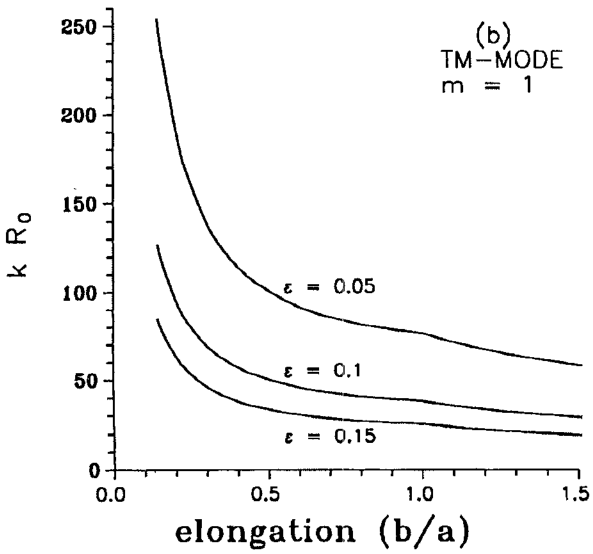
$$Ce_m(\eta_0, q) = 0 \text{ for the TM modes} \tag{14a}$$

$$Ce'_m(\eta_0, q) = 0 \text{ for the TE modes} \tag{14b}$$

when (8) is used to obtain the solutions, i.e., we have considered here only the even modes. When η_0 is fixed, the positive values of q, say $q_{m,l}$, for which the respective functions vanish are to be determined. These values of $q_{m,l}$ are regarded as the positive parametric zeros of the functions. For a given value of m, the Mathieu functions have an infinity of zeros with $l = 1, 2, 3 \cdots$. Also, for each value of l, m has an infinity of values. Hence there is a double infinity of

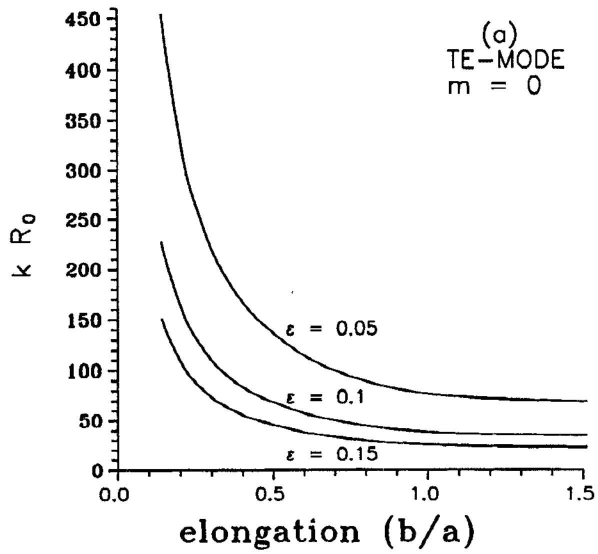


(a)

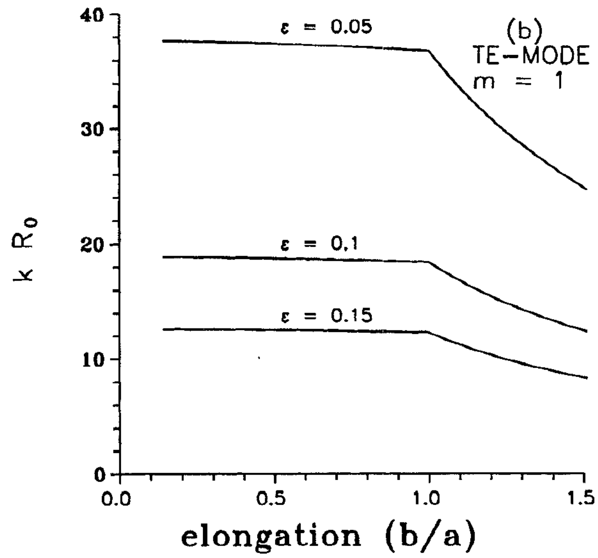


(b)

Fig. 2. Eigenfrequencies of TM modes plotted against elongation factor for different values of inverse aspect ratio. (a) $m = 0$. (b) $m = 1$.



(a)



(b)

Fig. 3. Eigenfrequencies of TE modes plotted against elongation for different values of inverse aspect ratio. (a) $m = 0$. (b) $m = 1$.

zeros. Equation (14) has been numerically solved to obtain the values of q and thereby the eigenfrequencies kR_0. The zeros of Mathieu functions are obtained by expanding the functions in a series in powers of q [12], [13].

The values of kR_0 have thus been obtained for the lowest zero of Mathieu function, i.e., $l = 1$, and for $m = 0, 1$. These have been plotted in Figs. 2 and 3 against elongation factor $e(= b/a)$ for different values of $\epsilon = 0.05, 0.1$, and 0.15 for both TM and TE modes. For the torus with a circular cross section, the resonant frequencies are obtained by evaluating the zeros of Bessel functions defined by (12).

The allowed eigenfrequencies for a elliptic torus decrease with increase in the elongation factor e and inverse aspect ratio ϵ. The effects of elongation as well as aspect ratio on the resonant frequencies are significant for both TE and TM modes.

The nonzero axial component of electric field for the TM modes is

$$E_\phi = \frac{C_m}{\sqrt{R_0 + \alpha \sinh \eta \cos \psi}} Ce_m(\eta, q) ce_m(\psi, -q)$$

for a torus with prolate elliptic cross section. The toroidicity of the configuration manifests itself in the appearance of the factor $\sqrt{R_0 + \alpha \sinh \eta \cos \psi}$, and the Mathieu functions bear resemblance to the eigen functions of elliptic resonators. The other field components can be obtained by substituting (8) in (3) and (4). The eigenmode solutions obtained here retain toroidicity as well as elongation effects even at the lowest order in perturbation expansion in terms of inverse aspect ratio.

V. Conclusion

The axisymmetric resonant modes in a toroidal cavity with elliptic cross section have been analyzed by assuming the inverse aspect ratio of the torus to be small. It is concluded that both the cross section elongation as well as inverse aspect ratio can have significant effects on the frequencies of the resonant cavity modes.

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BiCG-FFT T-Matrix Method for Solving for the Scattering Solution from Inhomogeneous Bodies

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***Abstract*—A BiCG-FFT T-Matrix algorithm is proposed to efficiently solve three-dimensional scattering problems of inhomogeneous bodies. The memory storage is of $O(N)$ (N is the number of unknowns) and each iteration in BiCG requires $O(N \log N)$ operations. A good agreement between the numerical and exact solutions is observed. The convergence rate for lossless and lossy bodies of various sizes are shown. It is also demonstrated that the matrix condition number for fine grids is the same as that for coarse grids.**

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I. Introduction

The scattering of electromagnetic fields by inhomogeneous bodies is a research topic that finds applications in many fields. In this paper, we propose a method of solving the inhomogeneous body problem by approximating the inhomogeneous body with small dielectric cubes. The dielectric cubes are then approximated by equivolume spheres [1]–[6]. In this manner, the T matrix [7], [8] of each individual sphere can be found in closed form. A set of linear algebraic equations can be easily derived to solve for the scattering amplitudes from each of the spheres. By using this T-matrix formulation, the Green's function singularity problem is avoided, while other formulations, such as the method of moments [9], such a singularity has to be handled with caution [10]–[12].

Direct solvers such as Gaussian elimination can be applied to solve for the scattering amplitudes in $O(N^3)$ operations and require $O(N^2)$ filling time of the matrix, where N is the number of unknowns. However, the computation is prohibitively intensive for large objects and the tremendous memory requirement cannot be met by most computers.

Iterative solvers such as CG (conjugate gradient) [13], [14] or BiCG (bi-conjugate gradient) method [15]–[17] can be used to circumvent the matrix storage difficulty although there are still $O(N^2)$ operations in each iteration and total number of iterations to converge is problem-dependent. In this work, we apply BiCG to solve for the solution of the matrix equation iteratively. When an iterative solver is used, the main cost of seeking the solution is the cost of performing a matrix-vector multiplication. When the inhomogeneous body is discretized into a regular grid, however, the resultant equation has a block-Toeplitz structure. Exploiting the block-Toeplitz structure, we can perform the matrix-vector multiplication in $O(N \log N)$ operations by FFT [10]–[12], [18].

The method can be shown to require $O(N)$ memory storage. Hence, it can be used to solve fairly large problems. A volume scattering problem with 90000 unknowns is solved on a Sparc 10 workstation. It is shown that iterative solvers converge faster for lossy bodies than lossless ones. This is because the matrix condition number for the former cases is smaller than that for the latter ones, as a lossless body could have high Q internal resonance modes.

As the simulation results show, by using the T-matrix formulation, the condition number of the resultant matrix is independent of the mesh size of a uniform grid. Therefore, the number of iterations does not grow when the body is gridded finer in order to achieve better resolution.

II. Formulation and Implementation

When a number of scatterers are placed on a uniform array, their scattering solution can be obtained efficiently by using FFT and an iterative method.

The total field due to an array of nonidentical scatterers can be written as

$$\mathbf{E}(\mathbf{r}) = \boldsymbol{\psi}^t(k_0, \mathbf{r}_s) \cdot \mathbf{a}_s + \sum_{i=1}^{N} \boldsymbol{\psi}^t(k_0, \mathbf{r}_i) \cdot \mathbf{b}_i \tag{1}$$

where $\mathbf{r}_i = \mathbf{r} - \mathbf{r}'_i$ and $\mathbf{r}'_i$ is the location of the scattering center of the ith scatterer. $\boldsymbol{\psi}^t(k_0, \mathbf{r}_i)$ is a row vector containing the vector spherical harmonics from each scatterer. The first term in (1) comprises the incident field while the second term is the scattered field. The vectors $\mathbf{a}_s$ and $\mathbf{b}_i$ contain the amplitudes of the incident